

STRICHARTZ ESTIMATES FOR THE PERIODIC NON ELLIPTIC SCHRÖDINGER EQUATION

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RÉSUMÉ. Le but de cette note est de démontrer des estimations de Strichartz optimales avec pertes de dérivées pour l'équation de Schrödinger non elliptique posée sur le tore de dimension 2.

ABSTRACT. The purpose of this note is to prove sharp Strichartz estimates with derivative losses for the non elliptic Schrödinger equation posed on the 2 dimensional torus.

Version française abrégée

On considère l'équation de Schrödinger non elliptique

$$(0.1) \quad (i\partial_t + \partial_x^2 - \partial_y^2)u = 0, \quad u(0, x, y) = u_0(x, y),$$

posée sur le tore de dimension deux $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. La solution de cette equation de donnée initiale u_0 est donnée par $e^{-itP}u_0$ où $P = -\partial_x^2 + \partial_y^2$. Un couple de nombres réels (p, q) est dit admissible si

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad p > 2.$$

Le théorème suivant donne des estimations de Strichartz optimales avec pertes de dérivées pour l'opérateur P .

Théorème 0.1. *Soit (p, q) un couple admissible. Il existe une constante $C > 0$ telle que pour toute donnée initiale $u_0 \in H^{\frac{1}{p}}(\mathbb{T}^2)$,*

$$(0.2) \quad \|e^{-itP}u_0\|_{L_{t \in [0,1]}^p L_{xy}^q(\mathbb{T}^2)} \leq C \|u_0\|_{H^{\frac{1}{p}}(\mathbb{T}^2)}.$$

De plus, l'estimation (0.2) est optimale au sens où l'inégalité

$$(0.3) \quad \|e^{-itP}u_0\|_{L_{t \in [0,1]}^p L_{xy}^q(\mathbb{T}^2)} \leq C \|u_0\|_{H^s(\mathbb{T}^2)}$$

devient fausse si $s < \frac{1}{p}$.

L'estimation de Strichartz (0.2) a été récemment démontrée dans le cas $p = q = 4$ (voir [4]) en utilisant une analyse globale sur le tore. La preuve du Théorème 0.1 que nous présentons est uniquement basée sur des arguments locaux. Le résultat du Théorème 0.1 peut être utilisé dans l'étude de perturbations non linéaires de (0.1). En particulier, l'analyse de [3] implique que dans le cas de perturbations cubiques, l'équation est bien posée dans H^s , $s > 1/2$. Des estimations de Strichartz avec pertes dans le cas elliptique

$$(0.4) \quad (i\partial_t + \partial_x^2 + \partial_y^2)u = 0, \quad u(0, x, y) = u_0(x, y),$$

posée sur \mathbb{T}^2 , ont été obtenues dans [1]. Pour l'équation (0.4), on ne connaît pas les estimations optimales pour tous les couples (p, q) admissibles (surtout pour $p < 4$) mais l'analyse de [1] montre que dans le cas $p = q = 4$, on a mieux que (0.2), à savoir que $1/p = 1/4$ peut être remplacé par n'importe quel nombre strictement positif.

1. INTRODUCTION

Consider the non elliptic Schrödinger equation

$$(1.1) \quad (i\partial_t + \partial_x^2 - \partial_y^2)u = 0, \quad u(0, x, y) = u_0(x, y),$$

posed on the two dimensional torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. The solution of (1.1) is given by $e^{-itP}(u_0)$, where $P = -\partial_x^2 + \partial_y^2$. We study here Strichartz estimates with losses for (1.1) and we show that the approach of [2] gives optimal estimates. We call a couple $(p, q) \in \mathbb{R}^2$ admissible if

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad p > 2.$$

We have the following statement.

Theorem 1.1. *Let (p, q) be an admissible couple. There exists a constant $C > 0$ such that for every $u_0 \in H^{\frac{1}{p}}(\mathbb{T}^2)$,*

$$(1.2) \quad \|e^{-itP}u_0\|_{L_{t \in [0,1]}^p L_{xy}^q(\mathbb{T}^2)} \leq C \|u_0\|_{H^{\frac{1}{p}}(\mathbb{T}^2)}.$$

Moreover (1.2) is sharp in the sense that the estimate

$$(1.3) \quad \|e^{-itP}u_0\|_{L_{t \in [0,1]}^p L_{xy}^q(\mathbb{T}^2)} \leq C \|u_0\|_{H^s(\mathbb{T}^2)}$$

fails for $s < \frac{1}{p}$.

The above result in the particular case $p = q = 4$ was recently obtained in [4] by using a different approach using the special choice of the L^4 norm and global analysis on the torus. The proof of Theorem 1.1 we present here relies only on local arguments.

The result of Theorem 1.1 can be used in the study of nonlinear perturbations of (1.1). In particular the analysis of [3] implies the well-posedness in H^s , $s > 1/2$ in the case of cubic perturbations.

Some Strichartz estimates with losses in the case of the elliptic Schrödinger equation

$$(1.4) \quad (i\partial_t + \partial_x^2 + \partial_y^2)u = 0, \quad u(0, x, y) = u_0(x, y),$$

posed on \mathbb{T}^2 , were obtained in [1]. In the context of (1.4), it seems that we do not have a clear picture what are the optimal Strichartz estimates for all admissible couples (p, q) (especially for $p < 4$). The analysis in [1] shows that in the particular case $p = q = 4$ one has better than (1.2), namely $1/p = 1/4$ can be replaced by every positive number which is almost the scale invariant estimate (the scale invariant estimate is however known to be false).

2. PROOF OF THEOREM 1.1

2.1. Proof of (1.2). Let $\Delta = \partial_x^2 + \partial_y^2$ be the Laplace operator. As in [2] (see [2, Corollary 2.3] and [2, second part of page 583]), by using the Littlewood-Paley square function theorem and the Minkowski inequality, in order to prove (1.2), it suffices to prove that for every $\varphi \in C_0^\infty(\mathbb{R})$, there exists $C > 0$ such that for every $h \in (0, 1]$, every $u_0 \in L^2(\mathbb{T}^2)$,

$$(2.1) \quad \|\varphi(h^2\Delta)e^{-itP}u_0\|_{L_{t \in [0,1]}^p L_{xy}^q(\mathbb{T}^2)} \leq Ch^{-\frac{1}{p}}\|u_0\|_{L^2(\mathbb{T}^2)}.$$

Let $\psi \in C_0^\infty(\mathbb{R})$ be such that $\psi(-x^2)\psi(-y^2)$ equals one on the support of $\varphi(-x^2 - y^2)$. Such a function exists since for a suitable $R > 1$ the support of $\varphi(-x^2 - y^2)$ is contained in the square $[-R, R] \times [-R, R]$ and thus it suffices to choose ψ which equals one on $[-R^2, R^2]$. Then

$$\psi(-x^2)\psi(-y^2)\varphi(-x^2 - y^2) = \varphi(-x^2 - y^2), \quad \forall (x, y) \in \mathbb{R}^2.$$

and hence

$$\psi(h^2\partial_x^2)\psi(h^2\partial_y^2)\varphi(h^2\Delta) = \varphi(h^2\Delta).$$

Therefore using the L^2 boundedness of $\varphi(h^2\Delta)$, we obtain that in order to get (2.1) it suffices to prove that for every $\psi \in C_0^\infty(\mathbb{R})$, there exists $C > 0$ such that for every $h \in (0, 1]$, every $u_0 \in L^2(\mathbb{T}^2)$,

$$(2.2) \quad \|\psi(h^2\partial_x^2)\psi(h^2\partial_y^2)e^{-itP}u_0\|_{L_{t \in [0,1]}^p L_{xy}^q(\mathbb{T}^2)} \leq Ch^{-\frac{1}{p}}\|u_0\|_{L^2(\mathbb{T}^2)}.$$

Let us denote by $K(t, x, y, x', y')$ the kernel of the map $\psi(h^2\partial_x^2)\psi(h^2\partial_y^2)e^{-itP}$, i.e.

$$(\psi(h^2\partial_x^2)\psi(h^2\partial_y^2)e^{-itP}u_0)(t, x, y) = \int_{\mathbb{T}^2} K(t, x, y, x', y')u_0(x', y')dx'dy'.$$

Then we have that

$$K(t, x, y, x', y') = K_1(t, x, x')K_2(t, y, y'),$$

where $K_1(t, x, x')$ is the kernel of $\psi(h^2 \partial_x^2) e^{it \partial_x^2}$ and $K_2(t, y, y')$ is the kernel of $\psi(h^2 \partial_y^2) e^{-it \partial_y^2}$. By [2, Lemma 2.5 and Remark 2.6], applied in the 1d case, we know that there exists $\alpha > 0$ such that

$$|K_1(t, x, x')| \leq C|t|^{-1/2}, \quad |K_2(t, y, y')| \leq C|t|^{-1/2}, \quad \forall |t| \leq \alpha h.$$

Consequently

$$|K(t, x, y, x', y')| \leq C|t|^{-1}, \quad \forall |t| \leq \alpha h.$$

Thus we obtain that there exists $C > 0$ such that for every $|t| \leq \alpha h$, every $u_0 \in L^1(\mathbb{T}^2)$,

$$(2.3) \quad \|\psi(h^2 \partial_x^2) \psi(h^2 \partial_y^2) e^{-itP} u_0\|_{L_{xy}^\infty(\mathbb{T}^2)} \leq C|t|^{-1} \|u_0\|_{L^1(\mathbb{T}^2)}.$$

With (2.3) in hand we can complete the proof of (1.2) exactly as in [2, page 583]. Indeed the $T - T^*$ argument implies that for every interval J of size $|J| \leq \alpha h$ one has

$$(2.4) \quad \int_J \|\psi(h^2 \partial_x^2) \psi(h^2 \partial_y^2) e^{-itP} u_0\|_{L^q(\mathbb{T}^2)}^p dt \leq C \|u_0\|_{L^2(\mathbb{T}^2)}^p.$$

Next we cover $[0, 1]$ with N intervals of size $\leq \alpha h$, $N \sim h^{-1}$ and using N times (2.4) we infer that

$$\begin{aligned} \int_0^1 \|\psi(h^2 \partial_x^2) \psi(h^2 \partial_y^2) e^{-itP} u_0\|_{L^q(\mathbb{T}^2)}^p dt &\leq \sum_{k=1}^N \int_{J_k} \|\psi(h^2 \partial_x^2) \psi(h^2 \partial_y^2) e^{-itP} u_0\|_{L^q(\mathbb{T}^2)}^p dt \\ &\leq C h^{-1} \|u_0\|_{L^2(\mathbb{T}^2)}^p \end{aligned}$$

This completes the proof of (1.2).

2.2. Optimality of the estimate. Let $f \in H^s(\mathbb{T})$. Then $f(x + y) \in H^s(\mathbb{T}^2)$ and $f(x + y)$ is a stationary solution of (1.1). Therefore if (1.3) holds then

$$(2.5) \quad \|f\|_{L^q(\mathbb{T})} \leq C \|f\|_{H^s(\mathbb{T})}.$$

Inequality (2.5) is the Sobolev embedding which is known to hold for $s \geq \frac{1}{2} - \frac{1}{q}$. It is also well known that it fails for $s < \frac{1}{2} - \frac{1}{q}$ as shows the next lemma.

Lemma 2.1. *Inequality (2.5) fails for $s < \frac{1}{2} - \frac{1}{q}$.*

Démonstration. It suffices to test (2.5) with

$$f(x) = \eta(\lambda x), \quad \lambda \geq 1, \quad \eta \in C_0^\infty(-1/2, 1/2).$$

We can see f as a $C^\infty(\mathbb{T})$ function and with this choice of f the left hand-side of (2.5) behaves like $\lambda^{-\frac{1}{q}}$ for $\lambda \gg 1$ while the right hand-side behaves like $\lambda^{s-\frac{1}{2}}$. Thus if (2.5) holds then $\lambda^{-\frac{1}{q}} \lesssim \lambda^{s-\frac{1}{2}}$ which implies $s \geq \frac{1}{2} - \frac{1}{q}$. \square

Using Lemma 2.1, we obtain that if (1.3) holds true then one should necessarily have

$$s \geq \frac{1}{2} - \frac{1}{q} = \frac{1}{p}$$

which proves the optimality of (1.2).

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